Basic Properties: ${}_U K_T$, ${}_T N_R$, ${}_R M_S$ bimodules.

(1) $R \otimes_R M \cong M$, $r \otimes m \mapsto rm$, $1 \otimes m \mapsfrom m$, and

$M \otimes_S S \cong M$ canonically

(2) $\left(\bigoplus_{i \in I} \overset{\in \text{Mod-}R}{N_i}\right) \otimes_R M \cong \bigoplus_{i \in I} (N_i \otimes_R M)$ (canonically)

$$\left[\left(\sum_{i \in I}' n_i\right) \otimes m \longmapsto \sum_{i \in I}' (n_i \otimes m) \right.$$

$$\left. \sum_{i \in I}' (n_i \otimes m_i) \longmapsfrom \sum_{i \in I}' (n_i \otimes m_i) \right]$$

(3) $(K \otimes_T N) \otimes_R M \cong K \otimes_T (N \otimes_R M)$ (canonically)

(4) If R commutative: $M \otimes_R N \cong N \otimes_R M$ (canonically)

As a consequence of (1),(2): if $F = \bigoplus_{i \in I} e_i R$ is free, $(F \cong R_R^{(I)})$

$$\underline{F \otimes_R M} \cong \bigoplus_{i \in I} e_i R \otimes_R M \cong \bigoplus_{i \in I} R \otimes_R M \cong \bigoplus_{i \in I} M = \underline{M^{(I)}}$$

<u>Lemma</u>: If F_R is free, $f: {}_R M \to {}_R N$ injective, then

$id_F \otimes f : F \otimes_R M \to F \otimes_R N$ is injective

<u>So</u>: $F \otimes_R -$ is exact; If R is a div. ring, all modules are free, so then tensor products are always exact.

<u>Proof</u>: $F = \bigoplus_{i \in I} e_i R$, each element of $F \otimes_R M$ has a repr. $\sum_{i \in I}' e_i \otimes m_i$

with <u>unique</u> m_i, and similarly for N.

$$0 = (id_F \otimes f)\left(\sum_{i \in I}' e_i \otimes m_i\right) = \sum_{i \in I}' e_i \otimes f(m_i) \implies \forall i: f(m_i) = 0 \Rightarrow m_i = 0.$$

$\square$

If $F = \bigoplus_{i\in I} e_i R$, $F' = \bigoplus_{j \in J} R f_j$ are free,

$$F \otimes_R F \cong \bigoplus_{i,j} \underbrace{e_i R \otimes_R R f_j}_{\cong R} \cong R^{(I\times J)} \quad \text{(as abelian group)}$$

If R commutative: $\rightarrow F \otimes_R F$ is free with basis $\{e_i \otimes e_j : i \in I, j \in J\}$

In particular: If $R = K$ field, V, W vector spaces $\Rightarrow \dim(V \otimes_K W) = \dim(V)\dim(W)$

6. Tensor Products of Algebras

k commutative ring, R, S k-algebras, $\otimes = \otimes_k$

[i.e. R ring & k-module s.t. $\lambda(rr') = (\lambda r)r' = r(\lambda r')$ $\forall r, r' \in R, \lambda \in k$

$\Leftrightarrow$ there is a ring hom $\varepsilon : k \to Z(R)$]

Prop 6.1: There is a unique k-algebra structure on $R \otimes S$ that satisfies

$$(r \otimes s)(r' \otimes s') = (rr') \otimes (ss') \qquad \forall r, r' \in R \ \forall s, s' \in S$$

Proof: Uniqueness: clear, since $R \otimes S = {}_k\langle r \otimes s : r \in R, s \in S\rangle$

Existence: Step 1: Fix r, s. Then $\begin{matrix} R \times S \longrightarrow R \otimes S \\ (r', s') \longmapsto rr' \otimes ss' \end{matrix}$ is k-bilinear

$\Rightarrow \exists \varphi_{r,s} : R \otimes S \longrightarrow R \otimes S$, $r' \otimes s' \longmapsto rr' \otimes ss'$. ($k$-hom.)

Step 2: $R \times S \longrightarrow \operatorname{Hom}_k(R \otimes S, R \otimes S)$, $(r,s) \longmapsto \varphi_{r,s}$ is k-bilinear

[e.g. $\forall r, r'' \in R, \lambda \in k$: $\varphi_{r+\lambda r'', s}(r' \otimes s') = (r + \lambda r'')r' \otimes ss'$

$= rr' \otimes ss' + \lambda(r''r' \otimes ss')$

$= \varphi_{r,s}(r' \otimes s') + \lambda \varphi_{r'',s}(r' \otimes s')$

$\Rightarrow \varphi_{r+\lambda r'', s} = \varphi_{r,s} + \lambda \varphi_{r'',s}$]

$\Rightarrow \exists \Phi : R\otimes S \to \mathrm{Hom}_k(R\otimes S, R\otimes S)$, $r\otimes s \mapsto \varphi_{r,s}$ (k-hom)

Now: $\mathrm{Hom}(R\otimes S, \mathrm{Hom}_k(R\otimes S, R\otimes S)) \cong \mathrm{Hom}((R\otimes S)\otimes(R\otimes S), R\otimes S)$

[$\otimes$-Hom adjunction], gives $(r\otimes s)\otimes(r'\otimes s') \mapsto \varphi_{r,s}(r'\otimes s') = rr'\otimes ss'$.

$\square$

Remark: (1) For $R \in k\text{-Alg}$, $R \cong k\otimes_k R$, $r\mapsto 1\otimes r$ and

$R \cong R\otimes_k k$, $r\mapsto r\otimes 1$ are k-algebra isomorphisms

(2) If R, S are k-algebras, $R \to R\otimes_k S$, $r\mapsto r\otimes 1_S$,

$S \mapsto R\otimes_k S$, $s\mapsto 1\otimes s$ are k-algebra hom.

(3) If R, S, T are k-algebras, $R\otimes_k S \cong S\otimes_k R$, $r\otimes s\mapsto s\otimes r$,

$(R\otimes_k S)\otimes_k T \cong R\otimes_k(S\otimes_k T)$, $(r\otimes s)\otimes t \mapsto r\otimes(s\otimes t)$

are k-algebra isomorphisms.

Observe: $(r\otimes 1)(1\otimes s) = r\otimes s = (1\otimes s)(1\otimes r)$, so in $R\otimes_k S$, the

images of R and S commute!

UP: If R, S are k-algebras, and $\varphi: R\to T$, $\psi: S\to T$ are

k-algebra hom s.t. $\forall r\in R\ \forall s\in S$: $\varphi(r)\psi(s) = \psi(s)\varphi(r)$, then

there exists a unique k-algebra hom $\gamma: R\otimes_k S \to T$ s.t.

$\forall r\in R\ \forall s\in S$: $\gamma(r\otimes s) = \varphi(r)\psi(s)$.

[By UP for tensor product of modules, there is a unique such k-hom γ,

because $(r,s)\mapsto \varphi(r)\psi(s)$ is k-bilinear. Observe that this map respects

multiplication.]

<u>Exm:</u> R k-algebra.

(1) $R \otimes_k M_n(k) \cong M_n(R)$

[E_{ij}, $1 \leq i,j \leq n$ is a k-basis of $M_n(k)$, so

$$R \otimes M_n(k) = R \otimes \bigoplus_{i,j=1}^{n} kE_{ij} \cong \bigoplus_{i,j=1}^{n} (R \otimes_k kE_{ij})$$

$\Rightarrow$ Elements of $R \otimes M_n(k)$ have a unique expression $\bigoplus_{i,j=1}^{n} r_{ij} \otimes E_{ij}$, $r_{ij} \in R$

Mapping $r_{ij} \otimes \underset{\in M_n(k)}{E_{ij}}$ to $r_{ij} \underset{\in M_n(R)}{E_{ij}}$ gives the isomorphism.

(2) $M_m(k) \otimes M_n(k) \cong M_n(M_m(k)) \cong M_{mn}(k)$

$$E_{ij} \otimes E_{st} \longmapsto E_{i+m(s-1),\, j+m(t-1)}.$$

$1 \leq i,j \leq m$, $1 \leq s,t \leq n$

<u>Extension of Scalars:</u>

k comm. ring, L comm. k-algebra.

·) Using $k \to L$ every L-algebra is a k-algebra (restrictions of scalars).

·) If R is a k-algebra, then $R \otimes_k L$ is an L-algebra (extension of scalars), since $(r \otimes 1)(1 \otimes \lambda) = r \otimes \lambda = (1 \otimes \lambda)(r \otimes 1)$.

<u>E.g.</u> $M_n(k) \otimes_k L \cong M_n(L)$, $\mathbb{Z}[x] \otimes_{\mathbb{Z}} k \cong k[x]$, $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$

<u>Prop 6.2:</u>

(1) If R is a k-algebra, S an L-algebra, then

$$\mathrm{Hom}_{L\text{-}Alg}(R \otimes_k L, S) \cong \mathrm{Hom}_{k\text{-}alg}(R, S)$$

(2) If R_1, R_2 are k-algebras, $(R_1 \otimes_k L) \otimes_L (R_2 \otimes_k L) \cong (R_1 \otimes_k R_2) \otimes_k L$

Proof: (1) By $\otimes$-Hom Adjunction,

$$\mathrm{Hom}_{L\text{-Mod}}(R\otimes_k L_L, S_L) \underset{(*)}{\cong} \mathrm{Hom}_{k\text{-Mod}}(R, \mathrm{Hom}_L(L_L, S_L))$$

S_L via $\varphi \mapsto \varphi(1)$

$$\underset{(**)}{\cong} \mathrm{Hom}_{k\text{-Mod}}(R, S)$$

via: $\psi \overset{(*)}{\longmapsto} (r \mapsto \psi_r)$, with $\psi_r(\ell) = \psi(r\otimes \ell)$

$(r \mapsto \varphi) \overset{(**)}{\longmapsto} (r \mapsto \varphi(1))$

together: $\psi \mapsto (r \mapsto \psi(r\otimes 1))$

This restricts to $\mathrm{Hom}_{L\text{-Alg}}(R\otimes_k L, S_L) \cong \mathrm{Hom}_{k\text{-Alg}}(R, S)$

Converse direction: given $g: R \to S$, $g\otimes id: R\otimes_k L \longrightarrow S$
$r\otimes \ell \mapsto g(r)\ell$

(2) $(R_1\otimes_k L)\otimes_L (R_2\otimes_k L) \cong (R_1\otimes_k L)\otimes_L (L\otimes_k R_2)$

$$\cong R_1\otimes_k (L\otimes_L L)\otimes_k R_2 \cong R_1\otimes_k L\otimes_k R_2 \cong (R_1\otimes_k R_2)\otimes_k L. \qquad \square$$

Bimodules again: R, S rings ($=\mathbb{Z}$-algebras)

$\{(R,S)\text{-bimodules}\} \cong \underline{\mathrm{Mod}}\text{-}R^{op}\otimes_{\mathbb{Z}} S$.

- If ${}_R M_S$ is a bimodule, there are structure homs $\varepsilon_R: R \to \mathrm{End}(M_{\mathbb{Z}})$, $\varepsilon_S: S^{op} \to \mathrm{End}(M_{\mathbb{Z}})$. Bimodule property: $\varepsilon_R(r)\varepsilon_S(s) = \varepsilon_S(s)\varepsilon_r(r)$ $\forall r\in R\ \forall s\in S$ $\quad (r(ms) = (rm)s)$

$\overset{\text{UP-}\otimes}{\Longrightarrow}$ $\exists$ unique ring hom $S^{op}\otimes_{\mathbb{Z}} R \to \mathrm{End}(M_{\mathbb{Z}})$, giving a

$\cong (S\otimes_{\mathbb{Z}} R^{op})^{op}$

right $R^{op}\otimes_{\mathbb{Z}} S$ module structure: $m(r\otimes s) = rms$

[More "pedestrian": $\mu_m: R^{op}\times S \longrightarrow M$, $(r,s)\mapsto rms$ is $\mathbb{Z}$-balanced, induces $(R^{op}\otimes S)\times M \to M$, $((r\otimes s), m) \mapsto rms$.]

- Other direction: $j_R: R^{op} \to R^{op} \otimes_{\mathbb{Z}} S$, $r \mapsto r \otimes 1$, $j_S: S \to R^{op} \otimes_{\mathbb{Z}} S$, $s \mapsto 1 \otimes s$.

If $M \in \underline{\text{Mod}} - R^{op} \otimes_{\mathbb{Z}} S$, restriction of scalars gives a left R-module, right S module structure. Images of j_R & j_S commute $\Rightarrow$ bimodule structure

$[\,(rm)s = (m(r\otimes 1))(1\otimes s) = m((r\otimes 1)(1\otimes s)) = m(r\otimes s) = m(1\otimes s)(r\otimes 1) =$

$\dots = r(ms).\,]$

6.1 Central Algebras

$\boxed{\text{Now: } k \text{ field},}$ $\otimes = \otimes_k$

If $0 \neq R$ is a k-algebra, the structure from $k \to Z(R)$ is injective.

Wlog $k \subseteq Z(R)$.

Def: A central k-algebra is a k-algebra $R \neq 0$ s.t. $Z(R) = k$.

Exm: $M_n(k)$ is central, $\mathbb{H}$ is a central $\mathbb{R}$-algebra, $\mathbb{C}$ is not a central $\mathbb{R}$-alg.
$M_n(D)$, D div. ring, is a $Z(D)$-central algebra.

Def: If R is a ring, $X \subseteq R$, the centralizer of X in R is

$$Z_R(X) := C_R(X) := \{ r \in R : \forall x \in X: rx = xr \}.$$

depends on author

- $Z_R(X)$ is a subring of R (k-subalgebra if R is a k-algebra).
- If R is a div. ring, so is $Z_R(X)$ $[\, rx = xr, r \neq 0 \Rightarrow xr^{-1} = r^{-1}x.\,]$

Thm 6.3: Let R, S be k-algebras. If $R' \subseteq R$, $S' \subseteq S$ are k-subalgebras, then $Z_{R\otimes S}(R' \otimes S') = Z_R(R') \otimes Z_S(S')$.

In particular:
- $Z(R \otimes S) = Z(R) \otimes Z(S)$
- If R, S central $\Rightarrow R \otimes S$ central

•) If R central, $L \supseteq k$ a field extension, then

$R \otimes_k L$ is a central L-algebra.

<u>Proof:</u> Since k is a field $R \otimes S$, $R' \otimes S'$, $Z_R(R') \otimes Z_S(S')$ are vector spaces, and $Z_R(R') \otimes Z_S(S')$, $R' \otimes S' \subseteq R \otimes S$ are wlog. subspaces, so the statement makes sense.

„$\supseteq$": If $r \in Z_R(R')$, $s \in Z_R(S')$, then, $\forall x \otimes y \in R' \otimes S'$,

$$(x \otimes y)(r \otimes s) = (xr) \otimes (ys) = (rx) \otimes (sy) = (r \otimes s)(x \otimes y)$$

$\Rightarrow r \otimes s \in Z_{R \otimes S}(R' \otimes S')$.

„$\subseteq$" Let $z \in Z_{R \otimes S}(R' \otimes S')$

$\Rightarrow z = \sum_{i=1}^{n} r_i \otimes s_i$ with $n \geq 0$, $r_i \in R$, $s_i \in S$.

Take n minimal among all such repr. of z.

Then $(r_1, \dots, r_n)$ is k-linearly independent, and so is $(s_1, \dots, s_n)$.

[A linear dependence would give $r_i = \sum_{j \neq i} \lambda_j r_j$ for some i, and give a shorter representation of z.]

Let $x \in S' \Rightarrow z(1 \otimes x) - (1 \otimes x)z = 0$

$$\Rightarrow \sum_{i=1}^{n} (r_i \otimes s_i)(1 \otimes x) - (1 \otimes x)(r_i \otimes s_i) = \sum_{i=1}^{n} r_i \otimes (s_i x - x s_i) = 0$$

$(r_i)_i$ k-lin. indep in $R \rightarrow (r_i)_i$ S-lin. indep in $R \otimes_k S$

$\Rightarrow \forall i: s_i x = x s_i \rightarrow s_i \in Z_S(S')$

Symmetrically: $r_i \in Z_R(R')$.

$\Big\} \rightarrow z \in Z_R(R') \otimes Z_S(S')$

$\square$

6.2 Simple Algebras

Again: $\boxed{k \text{ field},}$ $\otimes = \otimes_k$

A k-algebra R is simple if it is simple as ring.

Exm: $M_n(D)$, D div. ring, is a simple $Z(D)$-algebra; $A_n(k)$ (char $k = 0$)

R is central simple if it is central & simple.

[Note: R simple ring $\Rightarrow Z(R)$ is a field (easy exc.), so every simple ring is central simple over $Z(R)$; but we are fixing k.]

⚠ In many texts a central simple k-algebra (CSA) is central simple, and fin.-dim. $/k$!

Thm 6.4 Let R, S be k-algebras, S central simple. Then there is a bijection
$$\{\text{ideals of } R\} \longleftrightarrow \{\text{ideals of } R \otimes S\}$$
$$I \longmapsto I \otimes S$$
$$J \cap R \longleftarrow J$$

Lemma 6.5 If $0 \neq J \trianglelefteq R \otimes S$, then $J \cap R \neq 0$.

Proof: $R \to R \otimes S$, $r \mapsto r \otimes 1$ is injective (we work over a field), so wlog $R \subseteq R \otimes S$ and the claim makes sense.

Let $0 \neq x = \sum_{i=1}^{n} r_i \otimes s_i \in J$ with n minimal among all possible $x \in J \setminus \{0\}$.

$\Rightarrow (r_i)_i, (s_j)_j$ are each k-lin. independent.

$s_1 \neq 0 \Rightarrow S s_1 S = S \Rightarrow 1 = \sum_{i=1}^{m} x_i s_1 y_i \quad (x_i, y_i \in S)$.

Let $x' := \sum_{i=1}^{m} (1\otimes x_i)\, x\, (1\otimes y_i) = \sum_{i=1}^{m}\sum_{j=1}^{n} (1\otimes x_i)(r_j\otimes s_j)(1\otimes y_i)$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} r_j \otimes x_i s_j y_i = \sum_{j=1}^{n} r_j \otimes \underbrace{\left(\sum_{i=1}^{m} x_i s_j y_i\right)}_{=:\, s_j'} \quad , \quad s_1' = 1$$

Then $x' \in J$, and $x' \neq 0$ (the $(r_i)_i$ are k-lin. indep., and hence $(r_i\otimes 1)_i$ is S-lin. independent, and $s_1' \neq 0$)

$\forall s \in S$: $(1\otimes s)x' - x'(1\otimes s) = \sum_{j=1}^{n} (r_j \otimes s s_j' - r_j \otimes s_j' s)$

$$= \sum_{j=1}^{n} r_j \otimes (s s_j' - s_j' s) \underset{s_1'=1}{=} \sum_{j=2}^{n} r_j \otimes (s s_j' - s_j' s) \quad \in J$$

$\xRightarrow[\text{of } x]{\text{minimal choice}}$ $0 = \sum_{j=2}^{n} r_j \otimes (s s_j' - s_j' s) \Rightarrow \forall j:\ s s_j' = s_j' s$ (S-lin. indep. of $(r_j)_j$)

$\Rightarrow s_j' \in Z(S) = k \Rightarrow r_j \otimes s_j' = r_j s_j' \otimes 1$

$\Rightarrow x' = \sum_{j=1}^{n} (r_j s_j' \otimes 1) = \left(\sum_{j=1}^{n} r_j s_j'\right) \otimes 1 \in R \cap J.$ $\square$

<u>Proof of Thm 6.4:</u> Since $R \to R\otimes S$ is injective (again using that k is a field!), injectivity is clear.

<u>Show</u>: If $J \trianglelefteq R\otimes S$, then $J = (J\cap R)\otimes S$.

Let $I := J \cap R$. Then $I \otimes S \subseteq J$.

Consider $0 \to I \hookrightarrow R \to R/I \to 0$ (SES)

$\xRightarrow{(k \text{ field},\ S \text{ free})}$ $0 \to I\otimes S \to R\otimes S \xrightarrow{\pi} (R/I)\otimes S \to 0$ is a SES

$\Longrightarrow \ker(\pi) = I \otimes S$.

If $J \supsetneq I \otimes S$, $\quad 0 \neq \pi(J) \trianglelefteq (R/I) \otimes S \overset{[L6.5]}{\Longrightarrow} \pi(J) \cap R/I \neq \underline{0}$

i.e. $\exists r \in R \setminus I :\ (r+I) \otimes 1 \in \pi(J) \Rightarrow r \otimes 1 \in J \setminus I \otimes S$ ↯ $I = J \cap R$.

$\square$